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Existence of a Saddle Point in Nonconvex Constrained Optimization ⋆

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Abstract. The existence of a saddle point in nonconvex constrained optimization problems is considered in this paper. We show that, under some mild conditions, the existence of a saddle point can be ensured in an equivalent p-th power formulation for a general class of nonconvex constrained optimization problems. This result expands considerably the class of optimization problems where a saddle point exists and thus enlarges the family of nonconvex problems that can be solved by dual-search methods.

Key words: Nonconvex constrained optimization, Saddle point, Dual method, p-th power formulation, Global solution

1. Introduction

Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ and $g_j : \mathbb{R}^n \mapsto \mathbb{R}$, $j = 1, \dots, m$, be twice differentiable functions. Consider the following constrained optimization problem:

$$\min \quad f(x) \tag{1a}$$
$$\text{s. t.} \quad g_j(x) \leqslant b_j, \quad j = 1, 2, \dots, m, \tag{1b}$$
$$x \in X, \tag{1c}$$

where X is a closed and bounded subset of $\mathbb{R}^n$. Without loss of generality, we assume that f and g_js are strictly positive over X and $b_j > 0$ for all j. This assumption can be always satisfied via some equivalent transformations (e.g., exponential transformation) on (1).

It is well known that the saddle point condition is a sufficient condition for optimality. A crucial subject in constrained optimization is then the existence of a saddle point of the Lagrangian associated with problem (1). If there exists a saddle point, some efficient dual-search methods (see Luenberger, 1984) can be adopted to solve problem (1). In convex situations, the existence of a saddle point

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is proven in Karlin (1959) using the separation theorem under some constraint qualification conditions. The existence of a saddle point, however, is not guaranteed for nonconvex constrained optimization problems.

The existence of a saddle point is closely related to the convexity of the perturbation function of problem (1) defined by

$$w(y) = \min\{f(x) \mid g_j(x) \leqslant y_j,\ j = 1, 2, \ldots, m\ , x \in X\}, \tag{2}$$

The domain of the function w is

$$Y = \{y \in \mathbb{R}^m \mid \text{ there exists a } x \in X \text{ satisfying } g_j(x) \leq y_j,\ j = 1, 2, \ldots, m\}. \tag{3}$$

It has been shown that problem (1) possesses a saddle point if and only if the graph of w has a supporting hyperplane at $b = (b_1, \ldots, b_m)^T$ (see Minoux, 1986).

Applying a p-th power, with $p \geq 1$, to both the objective function and the constraints of (1) results in the following p-th power formulation (see Li, 1995 and Li, 1997) which is equivalent to problem (1),

$$\min \quad [f(x)]^p \tag{4a}$$

$$\text{s. t.} \quad [g_j(x)]^p \leq b_j^p, \quad j = 1, 2, \ldots, m, \tag{4b}$$

$$x \in X. \tag{4c}$$

Let $w_p(z)$ denote the perturbation function of the p-th power formulation in (4),

$$w_p(z) = \min\{[f(x)]^p \mid [g_j(x)]^p \leq z_j, j = 1, 2, \ldots, m, x \in X\}$$

Note that $w_p(y^p) = [w(y)]^p$. The domain of w_p is

$$Y_p = \{z = (y_1^p, \ldots, y_m^p)^T \mid y \in Y\}.$$

It has been proven in Li (1995) that under certain conditions the perturbation function of (4), w_p, is a convex function of z in a neighborhood of $b^p = (b_1^p, \ldots, b_m^p)^T$ when p is chosen sufficiently large. Thus a local saddle point is guaranteed to exist for the p-th power formulation. In many cases, this local saddle point is expected to be a global saddle point of problem (4). It has been shown in Xu (1997) that the same result can be obtained under weaker conditions by applying the p-th power transformation only to the constraints of (1).

The aim of this paper is to show that under some mild conditions, the local saddle point produced by the p-th power method is actually a global saddle point of problem (4) when p is chosen sufficiently large. The result in this paper expands considerably the class of optimization problems where a saddle point exists. Dual-based algorithms can then succeed in solving a general class of nonconvex problems in their p-th power formulations while the classical Lagrangian method fails due to a duality gap.

2. Existence of a Saddle Point

Let x^* be a global optimal solution of problem (1). We make the following conventional assumption about x^*.

ASSUMPTION 2.1

(a) There is a multiplier vector $\lambda^* \in \mathbb{R}_+^m$ satisfying the following first-order optimality conditions:

$$\left[\nabla f(x^*) + \sum_{j=1}^{m} \lambda_j^* \nabla g_j(x^*)\right]^T d \geq 0, \quad \forall d \in T(x^*), \tag{5}$$

$$\sum_{j=1}^{m} \lambda_j^* [g_j(x^*) - b_j] = 0, \tag{6}$$

where $T(x^*)$ denotes the tangent cone of X at x^*.

(b) The Hessian of the Lagrangian of problem (1):

$$H(x^*) = \nabla^2 f(x^*) + \sum_{j \in J(x^*)} \lambda_j^* \nabla^2 g_j(x^*)$$

is positive definite on $M(x^*)$, where

$$J(x^*) = \{j \in \{1, \ldots, m\} \mid \lambda_j^* > 0\},$$
$$N(x^*) = \{d \in \mathbb{R}^n \mid \nabla f(x^*)^T d = 0 \text{ and } \nabla g_j(x^*)^T d = 0,\ j \in J(x^*)\},$$
$$M(x^*) = N(x^*) \cap T(x^*).$$

(c) The set X is locally convex around x^*.

The associated Lagrangian function of the p-th power formulation in (4) is defined by

$$L_p(x, \mu) = [f(x)]^p + \sum_{j=1}^{m} \mu_j \{[g_j(x)]^p - b_j^p\}. \tag{7}$$

Let

$$\mu_p^* = [f(x^*)]^{p-1} (\lambda_1^*/[g_1(x^*)]^{p-1}, \ldots, \lambda_m^*/[g_m(x^*)]^{p-1})^T. \tag{8}$$

In the sequel, we always assume $J(x^*) \neq \emptyset$. Otherwise, (1) can be reduced to an unconstrained problem. The following theorem (see Xu, 1997, Theorem 2.3) shows that (x^*, μ_p^*) is a local saddle point of the Lagrangian defined by (7) when p is sufficiently large.

THEOREM 2.1 *Let x^* be a global optimal solution satisfying Assumption 2.1. Then there exists a $q_0 > 0$ and $\delta > 0$ such that*

$$L_p(x^*, \mu) \le L_p(x^*, \mu_p^*) \le L_p(x, \mu_p^*) \tag{9}$$

holds for all $x \in N(x^*, \delta) \cap X$ and $\mu \ge 0$ when $p \ge q_0$, where

$$N(x^*, \delta) = \{x \in \mathbb{R}^n \mid \|x - x^*\| \le \delta\}.$$

We now proceed to prove that under certain conditions inequality (9) holds for all $x \in X$ when p is chosen large enough. It is easy to see that the first inequality in (9) always holds by noticing condition (6) and the feasibility of x^*. We will prove in the following the second inequality in (9). Let $J^-(x^*) = \{1, \ldots, m\} \setminus J(x^*)$. Define

$$F = \{x \in X \mid g_j(x) \le b_j,\ \ j = 1, \ldots, m\}, \tag{10}$$

$$F_1(\epsilon) = \{x \in X \mid g_j(x) \le b_j + \epsilon,\ \ j \in J(x^*)\}, \tag{11}$$

$$F_2 = \{x \in X \mid g_j(x) \le b_j,\ \ j \in J^-(x^*)\}, \tag{12}$$

$$U(\epsilon) = \{x \in X \mid f(x) \le f(x^*) + \epsilon\}. \tag{13}$$

THEOREM 2.2 *Let x^* be a global optimal solution satisfying Assumption 2.1. Suppose that the following conditions hold:*

(i) *x^* is the unique global solution of (1);*
(ii) *If $J^-(x^*) \ne \emptyset$, then there exist positive ϵ_0 and ϵ_1 such that*

$$f(x) > f(x^*) + \epsilon_0, \quad \forall x \in F_1(\epsilon_1) \setminus F_2. \tag{14}$$

Then there exists a $q > 0$ such that (9) holds for all $x \in X$ and $\mu \ge 0$ when $p \ge q$.

Proof. Denote $N^0(x^*, \delta) = \{x \in \mathbb{R}^n \mid \|x - x^*\| < \delta\}$. Let

$$\begin{aligned} c &= \text{dist}\,(F, U(0) \setminus N^0(x^*, \delta)) \\ &= \min\{\|x - y\| \mid x \in F,\ \ y \in U(0) \setminus N^0(x^*, \delta)\}. \end{aligned}$$

We claim that $c > 0$. Otherwise, if $c = 0$, then there exist two sequences $\{x_n\}$ and $\{y_n\}$ satisfying $x_n \in F$, $y_n \in U(0) \setminus N^0(x^*, \delta)$ such that $x_n - y_n \to 0$. Since F is compact, we have $x_n \to \bar{x} \in F$ and $y_n \to \bar{y} \in U(0) \setminus N^0(x^*, \delta)$. It follows that $\bar{x} = \bar{y}$. Hence $\bar{x}$ is an optimal solution of problem (1) and $\bar{x} \ne x^*$, a contradiction to the uniqueness of the global optimum solution.

Define

$$\tilde{F}_2 = \begin{cases} F_2, & \text{if } J^-(x^*) \ne \emptyset \\ X, & \text{if } J^-(x^*) = \emptyset. \end{cases} \tag{15}$$

Note from (10)–(12) that $F = F_1(0) \cap \tilde{F}_2$. Thus, for any positive ϵ, the set $F_1(\epsilon) \cap \tilde{F}_2$ is an enlargement of the feasible region F by relaxing the strictly active constraints (with indices $j \in J(x^*)$). Since $c > 0$, by the compactness of F and the continuity of f and g_js, there exists a positive $\epsilon_2 \le \epsilon_1$ satisfying

$$[F_1(\epsilon_2) \cap \tilde{F}_2] \cap [U(0) \setminus N^0(x^*, \delta)] = \emptyset, \tag{16}$$

Let

$$Q = [F_1(\epsilon_2) \cap \tilde{F}_2] \setminus N^0(x^*, \delta), \tag{17}$$
$$\epsilon_3 = \min\{f(x) - f(x^*) \mid x \in Q\}. \tag{18}$$

Equation (16) implies $\epsilon_3 > 0$.

Now we prove the second inequality in (9) by contradiction. Suppose that there exists a sequence $\{x_p\} \subset X$ with $p \to +\infty$ such that

$$L_p(x_p, \mu_p^*) < L_p(x^*, \mu_p^*). \tag{19}$$

Note that $g_j(x^*) = b_j$ for $j \in J(x^*)$. Thus, from (7) and (8), we can rewrite (19) as

$$\sum_{j \in J(x^*)} \lambda_j^* \{[g_j(x_p)/b_j]^{p-1} g_j(x_p) - b_j\} < f(x^*) - [f(x_p)/f(x^*)]^{p-1} f(x_p). \tag{20}$$

For convenience, we denote

$$K_1 = \sum_{j \in J(x^*)} \lambda_j^* \{[g_j(x_p)/b_j]^{p-1} g_j(x_p) - b_j\},$$
$$K_2 = f(x^*) - [f(x_p)/f(x^*)]^{p-1} f(x_p).$$

The following can be verified using (11), (12) and (17),

$$X = Q \cup [X \setminus (F_1(\epsilon_2) \cup N^0(x^*, \delta))] \cup [F_1(\epsilon_2) \setminus (\tilde{F}_2 \cup N^0(x^*, \delta))] \cup [N(x^*, \delta) \cap X].$$

Notice from (15) that the set $F_1(\epsilon_2) \setminus (\tilde{F}_2 \cup N^0(x^*, \delta)) = \emptyset$ if $J^-(x^*) = \emptyset$. Since, based on Theorem 2.1, (20) does not hold for $x_p \in N(x^*, \delta) \cap X$ when p is sufficiently large, we only need to consider the following three cases.

Case (a): $x_p \in Q$. From (18), we have $f(x_p) \ge f(x^*) + \epsilon_3$. Thus

$$K_1 \ge -\sum_{j \in J(x^*)} \lambda_j^* b_j, \tag{21}$$
$$K_2 \le f(x^*) - [1 + \epsilon_3/f(x^*)]^{p-1} \underline{f}, \tag{22}$$

where $\underline{f} = \min_{x \in X} f(x) > 0$. Since $\epsilon_3 > 0$, letting $p \to +\infty$, (22) implies $K_2 \to -\infty$, which contradicts (20), when combined with (21).

Case (b): $x_p \in X \setminus (F_1(\epsilon_2) \cup N^0(x^*, \delta))$. By (11), there exists a $j_0 \in J(x^*)$ such that $g_{j_0}(x_p) > b_{j_0} + \epsilon_2$. Thus

$$K_1 > \lambda^*_{j_0}(1 + \epsilon_2/b_{j_0})^{p-1}(b_{j_0} + \epsilon_2) - \sum_{j \in J(x^*)} \lambda^*_j b_j, \tag{23}$$

$$K_2 \leq f(x^*). \tag{24}$$

Equation (23) implies that $K_1 \to +\infty$ as $p \to +\infty$, which contradicts (20), when combined with (24).

Case (c): $J^-(x^*) \neq \emptyset$ and $x_p \in F_1(\epsilon_2) \setminus (\tilde{F}_2 \cup N^0(x^*, \delta))$. Since $\epsilon_2 \leq \epsilon_1$, by (11) and (15), we have

$$x_p \in F_1(\epsilon_2) \setminus (F_2 \cup N^0(x^*, \delta)) \subset F_1(\epsilon_1) \setminus (F_2 \cup N^0(x^*, \delta)).$$

Thus, we have $f(x_p) > f(x^*) + \epsilon_0$ from (14). Using similar arguments as in case (a), we can derive a contradiction to (20).

PROPOSITION 2.1 *If the following holds:*

$$F_1(0) \cap U(0) = \{x^*\}, \tag{25}$$

then conditions (i) and (ii) in Theorem 2.2 are satisfied.

Proof. Since $F \subseteq F_1(0)$, condition (i) can be deduced from (25) directly. Now suppose that $J^-(x^*) \neq \emptyset$. By the assumption, we have

$$\text{dist}(F_1(0) \setminus F_2, U(0)) = \inf\{\|x - y\| \mid x \in F_1(0) \setminus F_2,\ y \in U(0)\} > 0.$$

Hence, there must exist ϵ_0 and ϵ_1 such that

$$[F_1(\epsilon_1) \setminus F_2] \cap U(\epsilon_0) = \emptyset,$$

which implies

$$f(x) > f(x^*) + \epsilon_0, \quad \forall x \in F_1(\epsilon_1) \setminus F_2.$$

Therefore, (14) is satisfied.

REMARK 2.1 Geometrically, condition (ii) in Theorem 2.2 requires that the contour $f(x) = f(x^*)$ does not extend to the area near the boundary of F_2. Thus, if there exists a supporting hyperplane separating $F_1(0)$ from $U(0)$ at x^*, then (14) will be satisfied. In the following, we verify (25) for convex programming under assumptions that x^* is a unique global solution of (1) and that there is no degenerate active constraints at x^*. Note that $F \cap U(0) = \{x^*\}$. Thus, by the convexity of f and g_js, there is a hyperplane separating F from U_0 at x^*, i.e., $\exists\lambda \neq 0$ such that

$$\lambda^T(x - x^*) < 0, \quad \forall x \in F,\ x \neq x^*, \tag{26}$$

$$\lambda^T(x - x^*) > 0, \quad \forall x \in U(0),\ x \neq x^*. \tag{27}$$

Since no active constraint is degenerate, $J(x^*)$ includes all the active indices of the active constraints, and hence there is a positive ϵ such that

$$F \cap N(x^*, \epsilon) = F_1(0) \cap N(x^*, \epsilon). \tag{28}$$

For any $x \in F_1(0)$, $x \neq x^*$, by the convexity of $F_1(0)$, we have $x^* + \alpha(x - x^*) \in F_1(0)$ for all $\alpha \in [0, 1]$. There must exist an $\alpha_0 \in (0, 1)$ such that $x^* + \alpha_0(x - x^*) \in F_1(0) \cap N(x^*, \epsilon)$. Hence $x^* + \alpha_0(x - x^*) \in F$ by (28). It follows from (26) that

$$\alpha_0 \lambda^T (x - x^*) = \lambda^T [x^* + \alpha_0(x - x^*) - x^*] < 0,$$

which implies $x \notin U(0)$ by (27). Therefore $F_1(0) \cap U(0) = \{x^*\}$.

From the equivalence between the existence of a saddle point and the existence of a supporting hyperplane of the perturbation function, we have the following corollary.

COROLLARY 2.1 *Under the conditions of Theorem 2.2, there exist $q > 0$ and for each $p > q$ a multiplier $\mu_p^* \in R_+^m$ satisfying*

$$w_p(z) \geq w_p(b^p) - (\mu_p^*)^T (z - b^p)$$

for all $z \in Y_p$ when $p \geq q$.

3. Illustrative Example

EXAMPLE 3.1 Consider the following constrained optimization problem:

$$\min \quad f(x) = 1 + (2x_1 - 3)(2x_2 - 3) \tag{29a}$$
$$\text{s. t.} \quad g_1(x) = 2x_1 - x_2 + 2 \leq 3, \tag{29b}$$
$$g_2(x) = x_2 + 1 \leq 2, \tag{29c}$$
$$x \in X = [0, 1.5]^2. \tag{29d}$$

This example is an indefinite quadratic problem and has a unique global optimal solution $x^* = (1, 1)^T$ with $\lambda^* = (1, 3)^T$, see Figure 1. The perturbation function defined in (2) is given as follows for this example problem:

$$w(y) = \begin{cases} 1 + (y_1 + y_2 - 6)(2y_2 - 5), & 3 \leq y_1 + y_2 \leq 6 \text{ and } 1 \leq y_2 \leq 2.5, \\ 1, & \text{otherwise.} \end{cases}$$

In the neighborhood of $y = (3, 2)^T$, $w(y)$ is an indefinite quadratic function with eigenvalues: $2 + 2\sqrt{2}$ and $2 - 2\sqrt{2}$. There is no supporting hyperplane at $y = (3, 2)^T$ and hence no saddle point for problem (29), see Figure 2.

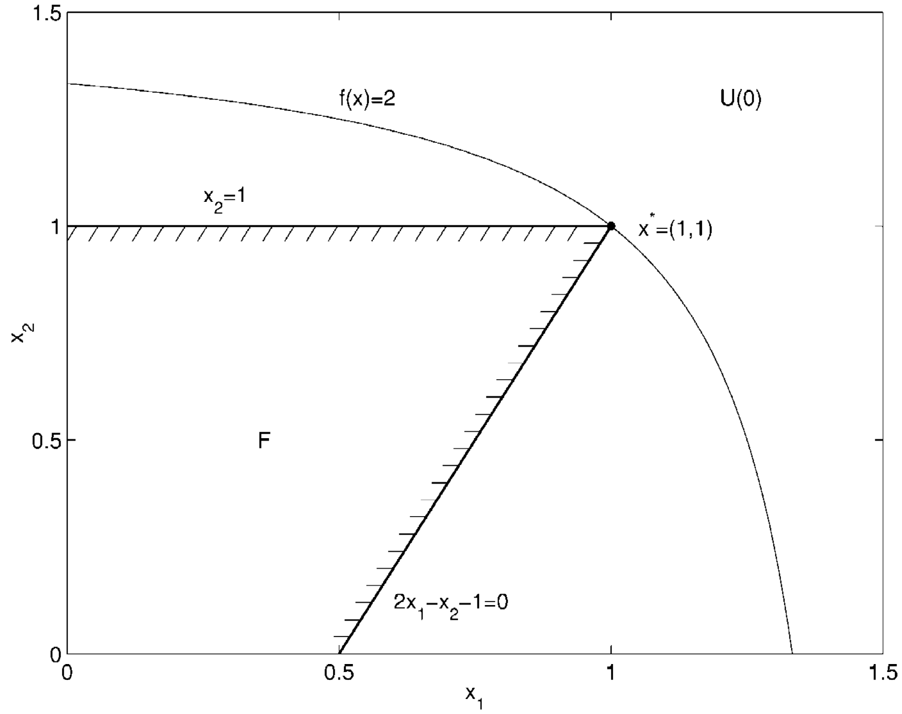

Figure 1. Geometrical illustration of Example 3.1.

Now consider the equivalent p-th power formulation of (29):

$$\begin{aligned}
\min \quad & [f(x)]^p = [1 + (2x_1 - 3)(2x_2 - 3)]^p && \text{(30a)}\\
\text{s. t.} \quad & [g_1(x)]^p = (2x_1 - x_2 + 2)^p \le 3^p, && \text{(30b)}\\
& [g_2(x)]^p = (x_2 + 1)^p \le 2^p, && \text{(30c)}\\
& x \in X = [0, 1.5]^2. && \text{(30d)}
\end{aligned}$$

It can be verified that Assumption 2.1 and conditions in Theorem 2.2 are satisfied at x^*. Hence there must exist a $p > 1$ and a multiplier $\mu_p^* \ge 0$ such that (x^*, μ_p^*) is a saddle point of (30). In fact, when $p = 3$, it can be verified that $\mu^* = \mu_3^* = (4/9, 3)^T$ is a saddle point multiplier for problem (30): $x^* = (1, 1)^T$ solves the following Lagrangian problem:

$$\min_{x \in X} L_3(x, \mu^*) = \min_{x \in [0,1.5]^2} \; [f(x)]^3 + 4/9\{[g_1(x)]^3 - 3^3\} + 3\{[g_2(x)]^3 - 2^3\}.$$

Combined with (6) and (8), this implies that (x^*, μ^*) is a saddle point of problem (30). Figure 3 illustrates the perturbation function $w_3(z)$ near $z = (27, 8)^T$.

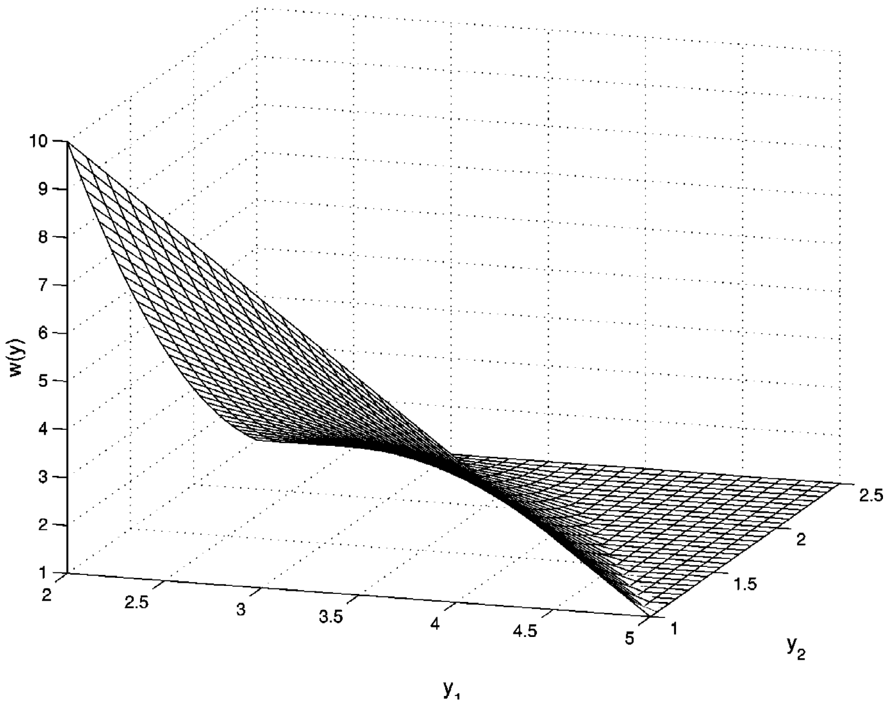

Figure 2. Perturbation function $w(y)$ of Example 3.1.

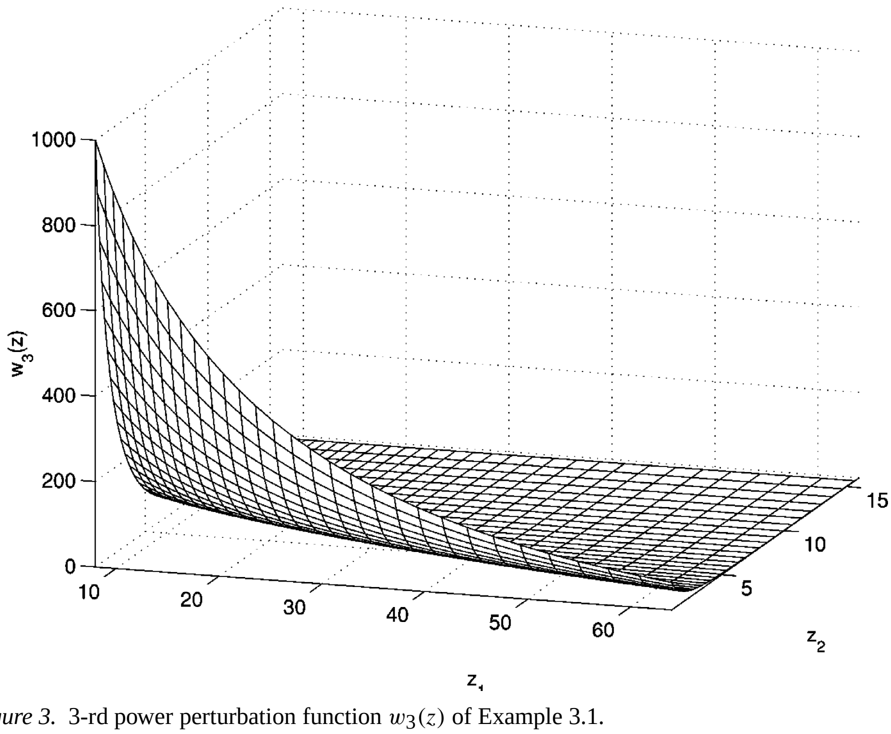

Figure 3. 3-rd power perturbation function $w_3(z)$ of Example 3.1.

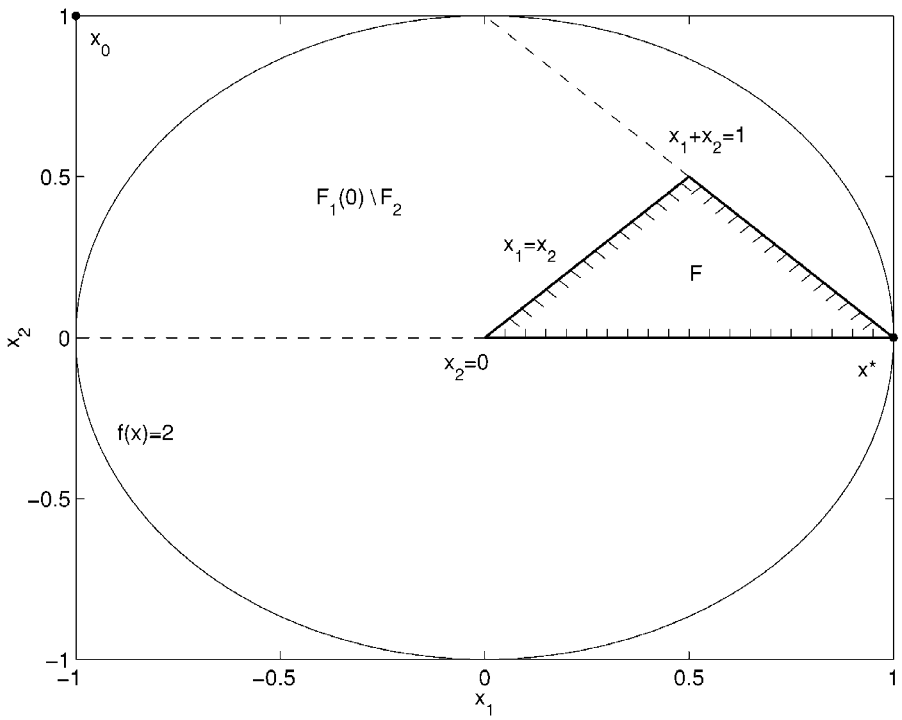

Figure 4. Illustration of condition (ii) in Theorem 2.2 for Example 4.1.

4. Final Remarks

The result in this paper extends the existence results of a saddle point from convex constrained optimization to a general class of nonconvex constrained optimization problems. The result on the existence of a saddle point in Theorem 2.2 can be further extended to certain extent.

Suppose there are finitely many global solutions of (1), $x^*_{[1]}, x^*_{[2]}, \ldots, x^*_{[K]}$. Let $g_j^{[k]} = g_j(x^*_{[k]})$, $k = 1, 2, \ldots, K$, $j = 1, 2, \ldots, m$. Suppose there exist no distinct $\hat{k}$ and $\tilde{k}$ such that $g_j^{[\hat{k}]} = g_j^{[\tilde{k}]}$, $\forall\ j = 1, ,2, \ldots, m$. Let $x^*_{[\hat{k}]}$ be a global solution of (1) such that there exists no other global solution $x^*_{[\tilde{k}]}$, $\tilde{k} \in \{1, 2, \ldots, K\}$, $\tilde{k} \neq \hat{k}$ with $g_j^{[\tilde{k}]} \leq g_j^{[\hat{k}]}$, $j = 1, 2, \ldots, m$, and at least one strict inequality holds. Then it can be concluded from Theorem 2.2 that a saddle point will exist for a p-th power formulation of the following modified version of (1), with $x^*_{[\hat{k}]}$ being the unique global optimal solution,

$$\begin{aligned} \min \quad & f(x) \\ \text{s. t.} \quad & g_j(x) \leq g_j^{[\hat{k}]}, \quad j = 1, 2, \ldots, m, \\ & x \in X, \end{aligned}$$

where the assumptions in Theorem 2.2 are assumed to be satisfied at $x^*_{[\hat{k}]}$.

The following example shows that condition (ii) in Theorem 2.2 is indispensable to ensure a saddle point to be produced by the p-th power Lagrangian formulation.

EXAMPLE 4.1 Consider a concave instance of problem (1):

$$\begin{aligned} \min \quad & f(x) = 3 - x_1^2 - x_2^2 && \text{(31a)} \\ \text{s. t.} \quad & g_1(x) = 3 - x_1 + x_2 \leq 3, && \text{(31b)} \\ & g_2(x) = 3 + x_1 + x_2 \leq 4, && \text{(31c)} \\ & g_3(x) = 2 - x_2 \leq 2, && \text{(31d)} \\ & x \in X = [-1, 1]^2. && \text{(31e)} \end{aligned}$$

A graphical illustration of this example is given in Figure 4. We see that the unique global optimal solution of (31) is $x^* = (1, 0)^T$. It can be verified that $\lambda^* = (0, 2, 2)^T$, $\mu_p^* = (0, 1/2^{p-2}, 2)^T$, $J(x^*) = \{2, 3\}$, $M(x^*) = \{0\}$. Consider point $x_0 = (-1, 1)^T \in X$. From Figure 4, we can see that $x_0 \in F_1(\epsilon) \setminus F_2$ for any $\epsilon > 0$. Also, $f(x_0) = 1 < 2 = f(x^*)$. Thus problem (31) does not satisfy condition (ii). We assert that the existence of a saddle point concluded in Theorem 2.2 is not ensured for this example. Indeed,

$$\begin{aligned} & L_p(x^*, \mu_p^*) = [f(x^*)]^p = 2^p, \\ & L_p(x_0, \mu_p^*) = 1^p + 1/2^{p-2} \times (3^p - 4^p) + 2 \times (1^p - 2^p) < 1. \end{aligned}$$

Hence, for any $p > 0$, we have $L_p(x^*, \mu_p^*) > L_p(x_0, \mu_p^*)$.

However, if we only take constraints g_2 and g_3 as the Lagrangian constraints while combining constraint g_1 with X, then a saddle point will exist in a p-th power formulation of the following problem,

$$\begin{aligned} \min \quad & f(x) = 3 - x_1^2 - x_2^2 && \text{(32a)} \\ \text{s. t.} \quad & \tilde{g}_1(x) = 3 + x_1 + x_2 \leq 4, && \text{(32b)} \\ & \tilde{g}_2(x) = 2 - x_2 \leq 2, && \text{(32c)} \\ & x \in \tilde{X} = \{x \in [-1, 1]^2 \text{ and } 3 - x_1 + x_2 \leq 3\} && \text{(32d)} \end{aligned}$$

Thus, the partitioning of the constraints between the set of Lagrangian constraints and the set of non-Lagrangian constraints has significant impact on the existence of a saddle point.

When $J^-(x^*) \neq \emptyset$, the perturbation function $w(y)$ is locally flat at $y = b$ along the coordinates y_i, $i \in J^-(x^*)$. From Example 4.1, we see that in order to ensure an existence of a saddle point, condition (ii) in Theorem 2.2 basically requires that the perturbation function $w(y)$ remain flat over the region: $\{y \in Y \mid y_j \leq b_i,\ j \in J(x^*),\ y_j > b_i,\ j \in J^-(x^*)\}$.

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